CONSTRUCTION OF A FIXED POINT FOR CONTRACTIONS IN BANACH SPACE

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ABSTRACT
A method for constructing fixed points of contractions in uniformly convex Banach spaces is developed. The fixed point obtained is the limit of one sequence that always converges (provided that a fixed point exists).

Introduction. Let X be a uniformly convex Banach space. Any contraction that takes a bounded closed convex set into itself has a fixed point (cf. Browder [1] and Kirk [4]). It is also known that a contraction has a fixed point if and only if the sequence of Picard iterates is bounded (cf. Kirk [4] and Browder and Petryshin [3]). A further result is that a fixed point of a contraction can be obtained as a limit of fixed points of strict contractions (cf. Browder [2]). Krasnoselskii [5] showed that if T is a contraction with range in a compact set then the definition $x_{n+1} = \frac{1}{2}(x_n + Tx_n)$ will result in a sequence converging to a fixed point of T. In this note we show in the non-compact case how to construct one sequence that always converges to a fixed point, provided that such a point exists. The construction is rather complicated.

It is possible to construct a complicated example showing that Krasnoselskii's construction will not do in the general case (J. Lindenstrauss, oral communication). Probably, a simple example is feasible and should be found.

Let X be uniformly convex. Denote $\alpha(t)$ by:

$$\alpha(t) = \tfrac{1}{2}\sup(2 - \| x + y \| ; \| x \| \leqq 1, \| y \| \leqq 1, \| x - y \| = t). \tag{1.1}$$

Then $\alpha(t) \to 0$ implies $t \to 0$.

For $x, y, z \in X$ denote $a = \max\{\| x - w \|, \| y - w \|\}$. Then clearly

$$\| \tfrac{1}{2}(x + y) - w \| \leqq a\left(1 - \alpha\left(\frac{\| x - y \|}{a}\right)\right). \tag{1.2}$$

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It is well known and easy to verify that $a \cdot \alpha((\| x - y \|)/a)$ is a monotone decreasing function of a. Indeed, consider points $x, y, z = \frac{1}{2}(x + y)$, w and w_1 where w_1 is on the segment joining z and w. Suppose also that $\| x - w \| \geqq \| y - w \|$. Then

$$\begin{aligned} \| x - w \| - \| z - w \| &\leqq \| x - w_1 \| + \| w - w_1 \| - (\| z - w_1 \| + \| w - w_1 \|) \\ &= \| x - w_1 \| - \| z - w_1 \| \leqq b - \| z - w_1 \| \end{aligned} \tag{1.3}$$

where $b = \max \{\| x - w_1 \|, \| y - w_1 \|\}$. The right hand side of (1.3) can be taken to be close to $b \cdot \alpha(\|x - y\| /b)$ while the left hand side is not smaller than $a.\ \alpha(\|x-y\|/a)$. Hence, $b \cdot \alpha((\| x - y \|)/b)$ is not smaller than $a \cdot \alpha((\| x - y\|)/a)$.

T is a contraction if $\| Tx - Ty \| \leqq \| x - y \|$ for $x, y \in D$. Let T be a contraction that takes some closed convex set C into itself. C is not necessarily bounded and may well coincide with X.

The construction below of a convergent sequence $\{y_n\}$ was motivated by the following reasoning. The definition $y_{n+1} = Ty_n$ does not result, in general, in a convergent sequence. Let us modify the process by defining, *for some* n $y_{n+1} = \frac{1}{2}(y_n + y_l)$ where $l < n$. Our aim is to reduce the distance to some unknown fixed point w. The gain is dependent on two contrasting factors: i) The distance between y_n and y_l. The larger the distance, the bigger the gain. ii) The differences between $\| y_n - w \|$ and $\| y_l - w \|$, the larger the difference the smaller the gain. Had we known the distances $\| y_n - w \|$ and $\| y_l - w \|$ we could have specified a rule of the following type: Take the combination $\frac{1}{2}(y_n + y_l)$ when $\| y_l - w \| - \| y_n - w \| = \delta$ and $\| y_n - y_l \| \geqq \varepsilon(\delta)$. Since the distances $\| y_n - w \|$ and $\| y_l - w \|$ are not known we devise a different measure to be constructed simultaneously with the sequence.

For that purpose, we choose a fixed sequence of positive numbers $\varepsilon_k \to 0$. Together with y_n we construct an integer valued infinite vector

$$(m_1(n), m_2(n), \cdots, m_k(n), \cdots)$$

where only a finite number of the components do not vanish. For such vectors define the sets of integers $S(k, n)$ as follows: $l \in S(k, n)$ if $l < n$ and $m_i(l) = m_i(n)$ for $i \leqq k$. The construction is inductive. Choose arbitrary point in the space, denote it by y_0 and set $m_i(0) = 0$ for all i. The passage from n to $n + 1$ is as follows.

Look at the first k for which either $S(k,n)$ is empty or for some $l \in S(k,n)$ $\| y_n - y_l \| \geqq \varepsilon_k$. (Since $\varepsilon_k \to 0$ such k exists). If $S(k,n)$ is empty define

$$y_{n+1} = Ty_n,\ m_i(n+1) = m_i(n) \qquad S(k,n) = \emptyset \tag{1.4}$$

Otherwise, define

$$y_{n+1} = \tfrac{1}{2}(y_n + y_l) \tag{1.5}$$

$$m_i(n+1) = m_i(n) \qquad i < k \tag{1.6}$$

$$m_k(n+1) = m_k(n) + 1 \tag{1.7}$$

$$m_i(n+1) = 0 \qquad i > k \tag{1.8}$$

The integer valued vector $m_i(n)$ introduces a lexicographic ordering of the integers n. Denote this ordering by $\cdot \gg \cdot$ (lec). It is easy to verify that this ordering is compatible with the natural ordering. Specifically:

$$n \geqq l \Leftrightarrow n \gg l(\text{lec}) \text{ or } n \equiv l \text{ (lec)} \tag{1.9}$$

$$n + 1 \equiv n\,(\text{lec}) \Leftrightarrow y_{n+1} = Ty_n \tag{1.10}$$

The lexicographic ordering according to the first k letters $m_i(n)$, $i \leqq k$ will be denoted by $\cdot \gg \cdot$ (lec, k). Also this ordering is compatible with the natural order. It has the properties:

$$n > l \Leftrightarrow n \gg l \text{ (lec}, k) \text{ or } n \equiv l \text{ (lec}, k) \tag{1.11}$$

$$n > l \text{ and } n \equiv l \text{ (lec}, k) \Leftrightarrow l \in S(k,n). \tag{1.12}$$

A consequence of the last relations is:

LEMMA 1. *$S(k,n)$ is composed of all the integers l: $l_0 \leqq l < n$. If $n-1 \notin S(k,n)$ then $S(k,n)$ is empty. If $j > k$ then $S(j,n) \subset S(k,n)$; in particular if $S(k,n)$ is empty so is $S(j,n)$.*

The fact that we can estimate the distance of y_n to a (unknown) fixed point w is expressed by the following lemma.

LEMMA 2. *Let w be an arbitrary fixed point of T. Let k be a fixed integer. Denote $\| y_0 - w \| = a$. Then $m_k(n) = r$ implies*

$$\| y_n - w \| \leqq a \left(1 - r \cdot \alpha\left(\frac{\varepsilon_k}{a}\right)\right). \tag{1.13}$$

Proof. By induction on n. (1.13) is certainly true for y_0. Suppose that it is true for all integers not exceeding n. Let $m_k(n+1) = r$. We have to investigate the following four possible cases.

i) $y_{n+1} = Ty_n, m_k(n+1) = m_k(n) = r$

ii) $y_{n+1} = \frac{1}{2}(y_n + y_l)$ and $m_k(n) = m_k(l) = r$

iii) $y_{n+1} = \frac{1}{2}(y_n + y_l)$ and $m_k(n) = m_k(l) = r - 1$

iv) $y_{n+1} = \frac{1}{2}(y_n + y_l)$ and $m_k(n+1) = 0$

For case i):

$$\| y_{n+1} - w \| = \| Ty_n - w \| = \| Ty_n - Tw \| \leqq \| y_n - w \|.$$

Since (1.13) holds for y_n it holds for y_{n+1} as well.
For case ii):

$$\| y_{n+1} - w \| \leqq \max\{\| y_n - w \|, \| y_l - w \|\} \tag{1.14}$$

establishing again (1.13).
For case iii) we use uniform convexity.

$$\| y_{n+1} - w \| \leqq b\left(1 - \alpha\left(\frac{\varepsilon_k}{b}\right)\right) \tag{1.15}$$

where $b = \max\{\| y_n - w \|, \| y_l - w \|\}$.

Since $b < a$ (by the induction hypothesis) it follows that

$$b \cdot \alpha\left(\frac{\varepsilon_k}{b}\right) \geqq a \cdot \alpha\left(\frac{\varepsilon_k}{a}\right). \tag{1.16}$$

The inductive hypothesis is:

$$b \leqq a\left(1 - (r-1)\alpha\left(\frac{\varepsilon_k}{a}\right)\right). \tag{1.17}$$

Substitute now (1.17) and (1.16) in (1.15) to get the desired estimate. In case iv) we use (1.14) and the fact that $r = 0$, establishing the lemma.

REMARK. $\| y_n - w \|$ is not necessarily a monotone function of n.

Our main result is:

THEOREM. *The sequence y_n defined in* (1.4) *or* (1.5) *converges to a fixed point of T if* (*and obviously only if*) *T has one.*

The theorem will be proved via the following three propositions.

PROPOSITION 1: *If T has a fixed point then $m_i(n)$ is constant for sufficiently large n* (*depending on i*).

PROPOSITION 2. *If, for all i, $m_i(n)$ is constant for sufficiently large n (depending on i) then $\{y_n\}$ converges.*

PROPOSITION 3: *If $\{y_n\}$ converges it converges to a fixed point of T.*

PROOF OF PROPOSITION 1. The left hand side of (1.13) is non-negative, therefore the $m_k(n)$ are bounded, precisely:

$$r = m_k(n) \leqq a\left[\alpha\left(\frac{\varepsilon_k}{a}\right)\right]^{-1} \tag{1.18}$$

The $m_k(n)$ are constructed so that either $m_k(n) \geqq m_k(n-1)$ or $m_k(n) = 0$. The last case holds when for some $l < k$: $m_l(n) = m_l(n-1) + 1$. Hence, the first of these integer valued functions: $m_1(n)$ is non-decreasing. Since, by (1.18) it is bounded it must be constant for $n \geqq n_1$. For $n \geqq n_1$ $m_2(n)$ will be non-decreasing and again, by (1.18) it must be constant for $n \geqq n_2$. Thus, inductively, the proposition is established for all k.

PROOF OF PROPOSITION 2. Let ε be given. Choose $\varepsilon_k < \varepsilon$ and consider n_k so large that $m_i(n)$ is constant for all $i \leqq k$ and $n \geqq n_k$. By the definition of $S(k,n)$ it follows that $l \in S(k,n)$ for $n > l > n_k$. The inequality $\| y_n - y_l \| \geqq \varepsilon_k$ implies now by the definition of $m_i(n)$ that for some $i \leqq k$ $m_i(n+1) > m_i(n)$ contrary to the assumption of constancy. Thus, $\| y_n - y_l \| < \varepsilon_k < \varepsilon$ for all $n, l: n > l > n_k$, i.e., Cauchy's criterion is satisfied.

PROOF OF PROPOSITION 3. Denote $y = \lim y_n$. We will show that for any index n there exists an r so that $y_{n+r+1} = Ty_{n+r}$. If this is the case then, clearly, $y = Ty$.

Now, if $y_{n+1} \neq Ty_n$ it means that, for some l_1 and k_1, $y_{n+1} = \frac{1}{2}(y_n + y_{l_1})$ where $l_1 \in S(k_1, n)$. Hence, $m_{k_1}(n+1) > m_{k_1}(n)$. Therefore, by Lemma 1 $S(k_1, n+1)$ is empty and, consequently, $S(j, n+1)$ is empty for $j > k_1$. If $y_{n+2} \neq Ty_{n+1}$ it means that, for some index k_2, there exists $l_2 \leqq n+1$ so that $l_2 \in S(k_2, n+1)$. Since $S(j, n+1)$ is empty for $j \geqq k_1$ it follows that $k_2 < k_1$. Continuing this way, we get, as long as $y_{n+i+1} \neq Ty_{n+i}$ sequences of integers $\{k_i\}$ and $\{l_i\}$ so that $l_i \in S(k_i, n+i)$. By the preceding argument the sequence $\{k_i\}$ is strictly decreasing; hence, it has only a finite number of terms. Thus, for some r, $y_{n+r+1} = Ty_{n+r}$. This concludes the proof of the proposition as well as that of the theorem.

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